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GAUSSIAN PROBABILITY MEASURE DEFINED BY
A GENERALIZED RELATIVE DENSITY

William C. Taylor

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Department of the Army Project No. 503-02-001
Ordnance Management Structure Code No. 5010.11.813
BALLISTIC RESEARCH LABORATORIES



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ABSTRACT

A generalized gaussian relative probability density is introduced in an infinite dimensional linear space. Although it is zero with probability one, it achieves to describe an actual probability distribution in another space by the behavior of its maxima on linear varieties. The calculus of random functions in terms of variational problems, which suggested this introduction, is to be the subject of subsequent reports.

An elementary property of gaussian probability densities in several variables is seen to motivate what must surely be the simplest, most direct and intuitive approach possible to gaussian probability distributions in infinite dimensional linear spaces. A gaussian distribution is described by means of a generalized relative probability density, the obvious formal generalization of the gaussian density in several variables. It cannot, of course, be the probability per unit volume since there is no volume. Rather the distribution is described in terms of the behavior of the maxima of this generalized density on linear varieties. The elementary property which motivates the treatment turns out to be just what is needed to fulfill the hypothesis of a fundamental theorem of Kolmogorov [1] which assures the existence, in any of certain larger spaces, of probability measures so described. The equivalence of these measures is then inferred without difficulty and perhaps contributes a bit to an understanding of the gaussian distribution.

The densities will be studied further and applications will be made in subsequent reports.

At this late date no introduction to gaussian distributions is apt to be totally new. Although the present treatment seems rather novel (to the author at least), it differs from many existing discussions principally in that it avoids the use of a preferred representation of the elements of the linear space. The maximum principle on which it is based has been used repeatedly by Feynman [1, 2] and probably others in the evaluation of individual functional integrals. It is pleasant to note that the space in which the density is defined need not be separable.

1. In the space R^N of N -tuplets $u: u_1, u_2, \dots, u_N$ of real numbers, a gaussian probability distribution is defined by a probability density

$$\varphi(u) = k \exp \left(-\frac{1}{2} J(u) \right) \quad (1)$$

where $J(u)$ is a positive definite quadratic polynomial, a translation of a positive definite quadratic form $K(u)$:

$$J(u) = K(u - u_0) \quad (2)$$

for some $u_0 \in R^N$. In (1), k is a normalizing constant which will henceforth be omitted. The resulting relative gaussian probability density will, for brevity, be described as a density. The values $v: v_1, \dots, v_M$, $M \leq N$, of any set of independent linear combinations of the u 's,

$$\ell(u): \ell_1(u) = \sum_{j=1}^N a_{1j} u_j, \quad i = 1, 2, \dots, M, \quad (3)$$

are random variables whose simultaneous distribution is described by a density $\psi(v)$ which may be found by integrating out the excess variables in (2). However, it may also be obtained simply as the maximum of $\varphi(u)$ taken subject to the condition that the linear functions $\ell(u)$ take the specified values:

Theorem 1 If u is a set of gaussian random variables having a density $\varphi(u)$, the density of the values v of independent linear combinations (3) is

$$\begin{aligned} \psi(v) &= \max (\varphi(u) : \ell(u) = v) \\ &= \exp \left(-\frac{1}{2} \min (J(u) : \ell(u) = v) \right). \end{aligned} \quad (4)$$

This result becomes apparent when a linear transformation is made to independent variables of which the first M are linear combinations of $\ell_1(u), \dots, \ell_M(u)$.

So far as the author knows, this simple property of normal distributions has not before been expressly stated. Here it will serve first to motivate the definition of a generalized density in an infinite space, and second, to provide a principal cog in the mechanism by which the generalized density is seen to describe a probability distribution in a different space, a certain second conjugate space.

2. Let S be a real linear space in which there is defined an inner product (u, v) , $u, v \in S$, which is positive definite, i.e.,

$$(u, u) = \|u\|^2 > 0, \text{ if } u \neq 0. \quad (5)$$

Such a space S may be completed to form the unique minimal hilbert space $\bar{S} \supset S$. A positive definite quadratic functional $J(u)$, $u \in S$, is a functional which can be written in terms of such an inner product as

$$J(u) = \|u\|^2 + \ell(u) + \text{const} \quad (6)$$

where $\ell(u)$ is a linear functional bounded with respect to $\|u\|$. The definition of the functionals $J(u)$ and $\ell(u)$ may be extended continuously to \bar{S} in a unique way. The representation theorem assures the existence of an element $u_0 \in \bar{S}$ such that

$$\ell(u) = -\frac{1}{2}(u_0, u) \quad (7)$$

for all $u \in \bar{S}$. Then, using the linearity of the inner product,

$$J(u) = \|u - u_0\|^2 + \text{const.} \quad (8)$$

A functional

$$\varphi(u) = \exp - \frac{1}{2} J(u), \quad u \in S, \quad (9)$$

where $J(u)$ is a positive definite quadratic functional will be called a generalized relative gaussian probability density, or briefly, a density. It will be the aim of the following paragraphs to justify this terminology.

Let S' be the normed conjugate space of S , consisting of all bounded linear functionals defined on S . To each $v'(u) \in S'$ there corresponds $v \in \bar{S}$ such that $v'(u) = (v, u)$ for all $u \in S$ (or \bar{S}). This correspondence, in which both norm and linear structure are invariant, makes it frequently unnecessary to distinguish between a hilbert space and its conjugate. Here, however, the reader may find it helpful to emphasize the distinction. Further, let $\hat{S} = \{\hat{u}\}$ be the algebraic conjugate space of S' , consisting of the linear functionals, $\hat{u}(u')$, not necessarily bounded, defined on S' . For these functionals the following extensions of earlier notation will be used:

$$\hat{u}(v') = v'(\hat{u}) = (v, \hat{u}). \quad (10)$$

The space \hat{S} will be the space in which is to be defined the probability measure described by (9). To each element v' of S' will then be associated a random variable (10).

Any finite set u_1', u_2', \dots, u_N' of linearly independent elements of S' has an N -dimensional probability distribution described by a density $\psi(w_1, w_2, \dots, w_N)$ in the ordinary

sense that if E is any measurable set of R^N , the probability that the N -tuple $U^N: u_1'(u), u_2'(u), \dots, u_N'(u)$ lies in E is

$$\int_E \psi(w) dw_1 \dots dw_N / \int_{R^N} \psi(w) dw_1 \dots dw_N. \quad (11)$$

The density is given in analogy with (4) by the conditioned supremum

$$\begin{aligned} \psi(w) &= \psi_{u_1', u_2', \dots, u_N'}(w_1, \dots, w_N) \\ &= \sup (\varphi(u): u \in S, u_1'(u) = w_1, i = 1, 2, \dots, N) \\ &= \exp - \frac{1}{2} \inf (J(u): u \in S, u_1'(u) = w_1, i = 1, 2, \dots, N). \end{aligned} \quad (12)$$

The supremum may, of course, be replaced by maximum if S is replaced by \bar{S} . In order to verify that $\psi(w)$, so defined is a gaussian density, it must be shown the exponent which occurs in (12) is a definite quadratic polynomial. There are N elements $v_1 \in \bar{S}$, $i = 1, 2, \dots, N$, by means of which it is possible to express the functionals u_1' as inner products:

$$u_1'(u) = (v_1, u) \text{ for all } u \in S. \quad (13)$$

Any element $u \in S$ can be expressed, with real a_1, \dots, a_N , in the form

$$u = u_0 + \sum_{i=1}^N a_i v_i + r, \quad (14)$$

where

$$r \in \bar{S} \text{ and } (v_i, r) = 0, i = 1, 2, \dots, N. \quad (15)$$

Then the conditions on the infimum in (12) can be written

$$w_j = (v_j, u) = (v_j, u_0) + \sum_{i=1}^N (v_j, v_i) a_i, \quad (16)$$

$$j=1, 2, \dots, N.$$

The statement that the v_1, \dots, v_N are linearly independent is equivalent to the non-vanishing of the determinant of the quantities (v_i, v_j) , $i, j = 1, \dots, N$; and it is seen from (16) that specifying (v_j, u) , $j = 1, \dots, N$, is equivalent to specifying a_1, \dots, a_N . Since

$$\begin{aligned} J(u) &= \|u - u_0\|^2 = \sum a_i v_i + \|r\|^2 \\ &= \sum_{i,j=1}^N (v_i, v_j) a_i a_j + \|r\|^2, \end{aligned} \quad (17)$$

it is clear that the infimum of $J(u)$ with the a 's specified is obtained by setting $r = 0$, and that the value so obtained is positive and is a quadratic form in a_1, \dots, a_N , or, equivalently, in w_1, \dots, w_N .

Finally, in order to justify the term generalized density, it must be shown that there exists a probability measure of which the array of marginal densities obtained from (12) is a proper description. This results from a straightforward application of the fundamental theorem of Kolmogorov [1, Ch. III]:

Let M be any set. Let R^M be the space of real functions x_μ defined before all $\mu \in M$. If for every finite subset $\mu_1, \mu_2, \dots, \mu_N$ of M there is given a probability distribution in the space R^N of

the values of $x_{\mu_1}, x_{\mu_2}, \dots, x_{\mu_N}$, and if these finite dimensional distributions are compatible, then there exists a countably additive probability measure in R^M defined, over the field of those subsets of R^M countably generated by half spaces (sets defined for some $\mu \in M$ and real k , by $x_\mu < k$) and having the given distributions as marginal distributions, i.e., as distributions of sets $x_{\mu_1}, \dots, x_{\mu_N}$ considered as random variables on R^M .

In applying this theorem to the present situation the index set M may be taken as a Hamel basis Z of S' , that is, a finitely linearly independent subset of S' such that every element of S' can be expressed in a unique way as a finite linear combination of elements of Z . The compatibility condition, that the given lower dimensional distributions are obtained in every case by integrating out the excess variables from the given higher dimensional distributions, is here immediately seen to be satisfied by virtue of Theorem 1 and the fact that, for $N_1 < N$,

$$\begin{aligned} \max_{k_{N_1+1}, \dots, k_N} \sup_{u \in S} (\varphi(u): u_i'(u) = k_i, i = 1, 2, \dots, N) \\ = \sup_{u \in S} (\varphi(u): u_i'(u) = k_i, i = 1, 2, \dots, N_1). \end{aligned} \quad (18)$$

The conclusion may now be drawn that the generalized density describes a probability measure on the space of functionals on Z . But since Z is a Hamel basis of S' , the linear functionals on S' are one-to-one linear images of the functionals on Z . The generalized density may thus equally well be said to describe a probability measure on \hat{S} . It is also clear that this measure is independent of the choice of the Hamel basis Z . This completes the proof of

Theorem 2: A generalized gaussian density (9) defined on a linear space S : $\{u\}$ describes by means of the marginal densities (12) a unique, countably additive probability measure on the space \hat{S} of linear functionals defined on the normed conjugate space S' of S , consisting of those linear functionals on S which are bounded with respect to the norm provided by the given density. This measure is defined over the borel field of sets of \hat{S} generated countably by half spaces, the sets $\{\hat{u} \in \hat{S} : \hat{u}(u') < k\}, u' \in S', k \text{ real.}$

3. Either as a consequence of Theorem 2, or more simply by repeating the argument with minor amendments the following corollary is obtained.

Corollary: Let V' be a subset of S' and \hat{V} the space of linear functionals defined on V' . The generalized density (9) describes a unique, countably additive probability measure in \hat{V} over the borel field of sets of \hat{V} generated by half spaces $\{\hat{v} \in \hat{V} : \hat{v}(v') < k\}, v' \in V', k \text{ real.}$

If V' is complete in S' , the measure defined in \hat{V} is an adequate representation of the measure defined in \hat{S} in the sense described in the following theorem. For the statement, there is needed the definition that two sets are equivalent if their differences are of measure zero.

Theorem 3: If the set V' is complete in S' , the measures defined on \hat{S} and \hat{V} are isomorphic in the sense that there is a one-to-one correspondence between classes of equivalent sets in the two spaces in which measure, inclusion and translation by elements of \bar{S} are preserved.

Proof: There is a projection of \hat{S} into \hat{V} in which each element of \hat{S} , a linear functional on S' , is carried into its restriction to V' . To each measurable set E_V of \hat{V} let correspond the set E_S of \hat{S} consisting of all elements which are projected into elements of E_V . A sequence of operations on the half spaces of \hat{V} which defines E_V defines similarly when applied to the corresponding half spaces of \hat{S} , a set of E_S of the same measure. There remains only the question whether every measurable set of S is included in this correspondence, and it will suffice to show that every half space is included, or, in other words, to show that to each bounded linear functional $u' \in S$ there is an equivalent random variable defined on \hat{V} . Since V' is complete in S' , there is a sequence v_1', v_2', \dots in V' such that $\|v_1' - u'\|$ tends to zero. An immediate consequence of (12) is that the norm of any bounded linear functional on S is the standard deviation of the random variable it defines. The sequence $\{v_i'\}$ must therefore be regular in probability in V and defines (see e.g., Halmos [1] Sec. 22, Theorem E, p. 93) in \hat{V} a random variable equivalent to u' (ibid., Theorem C). This completes the proof.

Perhaps it should be emphasized that this isomorphism is not a mapping of elements. If the space is infinite and if V' is a small (compared to a Hamel basis) set then it will not be possible to express any significant portion of the elements of \hat{S} as linear images of elements of \hat{V} . If S is a separable space, then, as Friedrichs and Shapiro [1, 2] have shown, linear mappings of elements can be produced which work with probability one and provide a realization of the isomorphism of the measure spaces related by a change of coordinates.

William C. Taylor
 WILLIAM C. TAYLOR

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